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Non-commutative spheres and the AdS/CFT correspondence

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Abstract: We present direct arguments for non-commutativity of spheres in the AdS/CFT correspondence. The discussion is based on results for the S_N orbifold SCFT. Concentrating on three point correlations (at finite N) we exhibit a comparison with correlations on a non-commutative sphere. In this manner an essential signature of non-commutativity is identified giving further support for the original proposal of hep-th/9902059.

Keywords: 1/N Expansion, AdS/CFT Correspondance, Non-Commutative Geometry.

Contents

1. Introduction

The AdS/CFT correspondence [1, 2, 3] provides a constructive approach to supergravity, and closed string theory in curved backgrounds. Its basis is the large-N expansion of Yang-Mills type theory which turns into a loop expansion of gravity. The existence of a deductive procedure has prompted questions concerning any modification that the emerging gravity could exhibit. In particular a proposal was made in [4, 5], that the curved SUGRA background $AdS \times S$ is non commutative with the non-commutativity parameter given by $1/N$. This follows an earlier proposal for q-deformation given in a framework of a simple matrix model [6]. The extension to higher spaces was further discussed in [7].

The non-commutativity of $AdS \times S$ space naturally incorporates the exclusion principle of [8] and stands to have important implications on the physics of black holes. It also conforms to a general principle of [9] that string/M-theory inherently contains a space-time uncertainty (see [10]). Recently, a physical argument for the non-commutativity and the associated cutoff was given in [11] based on a study of brane motions on spheres. This discussion involves a mechanism [12, 13] by which gravitons are polarized into extended spherical membranes which lead to non-commutativity [14]. Other examples are given in [15]–[19].

It is clearly important to further clarify the nature of non-commutativity in $AdS \times S$ spaces. In comparison with the non-commutativity induced by an external B-field for the origin of non-commutativity in $AdS \times S$ is less trivial to exhibit. Since it involves the closed string coupling $1/N$, it represents a non-perturbative phenomenon.

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In this paper, we describe further evidence for the above non-commutativity in AdS/CFT. The discussion is based on the S_N orbifold model that already served as the basis for arguments presented in [4, 5]. In this model, one is able to perform explicit calculations of three point interactions at finite N and study their behavior. In this way, we exhibit an explicit signature for non-commutativity of the corresponding spheres.

The content of the paper is as follows. In section 2, we summarize the finite-N results for three-point correlations in the orbifold CFT and discuss their properties. We then review the (super)gravity calculations in commutative space-time in section 3 and proceed to evaluate the modifications due to a non-commutative sphere in section 4. We exhibit certain agreement with the SCFT study.

2. Results from S_N orbifold

In this section, we review the results of CFT dual to the gravity in the case of $AdS_3 \times S^3$ obtained in [5]. We also use the extension of these results to the non-extremal case that can be read off from recent work of [20].

The SCFT in question is defined on symmetric product $S^N(M)$, where M is either $\mathbb{T}^4$ or $K3$. The field content of the theory consists of: $4N$ real free bosons $X_I^{a\dot{a}}$ representing the coordinates of the torus for example and their superpartners $4N$ the fermions $\Psi_I^{\alpha\dot{a}}$, where $I = 1, \ldots, N$, $\alpha, \dot{\alpha} = \pm$ are the spinorial S^3 indices, and $a, \dot{a} = 1, 2$ are the spinorial indices on $\mathbb{T}^4$. In essence, the field content of the theory is determined to be $4N$ real free bosons and $2N$ Dirac free fermions, giving a central charge $c = 6N$. One has left and right superconformal symmetry with the corresponding currents. The lowest modes of this currents, namely $\{L_{0,\pm 1}, G^{\alpha a}_{\pm 1/2}, J_0^{\alpha\beta}\}$ together with their right counterparts generate the $\mathrm{SU}(2|1,1)_L \times \mathrm{SU}(2|1,1)_R$ symmetry. These according to the AdS/CFT correspondence translate into the superisometries of the $AdS_3 \times S^3$ space-time. In addition, one has other symmetries commuting with the previous set for example the ones related to global $\mathbb{T}^4$ rotations. Even though the underlying CFT on $\mathbb{T}^4$ is very simple, the complexity is given by the non-trivial implementation of the S_N symmetry of the orbifold. This symmetry is an analogue of $\mathrm{U}(N)$ gauge symmetry in this context. Physical observables analogous to traces are now given by S_N invariants. In particular the complete set of chiral primary operators was given in [4, 5]. A fundamental role is played by the twist operators that impose the twisted the boundary conditions:

$$\begin{aligned} X_I\left(z\, e^{2\pi i}, \bar{z}\, e^{-2\pi i}\right) &= X_{I+1}(z, \bar{z})\,, \quad I = 1, \ldots, n-1\,, \\ X_n\left(z\, e^{2\pi i}, \bar{z}\, e^{-2\pi i}\right) &= X_1(z, \bar{z}) \end{aligned} \tag{2.1}$$

for the n-twisted sector of the theory. First a construction of Z_n twist operators is given in terms of which the S_N invariant chiral primary operators are constructed after appropriately averaging over S_N.

In the correspondence with gravity on $AdS_3 \times S^3$ one achieves a one-one correspondence with single particle states.

The computation of two and three point functions (for extremal momenta) was given in [5] and has recently been extended in [20]. We present the three-point function for chiral primaries $O_n^{(0,0)}$, where n denotes the length of the cycle (twist) used to construct the operator. The index n is also identified with l of the angular momentum on S^3 in gravity as $n = l + 1$ (let us recall that the isometry group for S^3 is $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and that l is an angular momentum in the diagonal $\mathrm{SU}(2)$). The correlation functions in terms of $n = l_1 + 1$, $k = l_2 + 1$ and $n + k - 1 = l_3 + 1$ for the case $l_3 = l_1 + l_2$ (extremal) are:

$$\left\langle O_{n+k-1}^{(0,0)\,\dagger}(\infty)\, O_k^{(0,0)}(1)\, O_n^{(0,0)}(0) \right\rangle = \left(\frac{(N-n)!\,(N-k)!\,(n+k-1)^3}{(N-(n+k-1))!\,N!\,n\,k} \right)^{1/2} . \tag{2.2}$$

This expressions for the three point correlation functions obtained in [5] contains two types of factors: one that is solely dependent on the angular momenta with no N dependence and the other with explicit N dependence. It is the latter type that we concentrate on as it contains direct implications on the non-commutative nature of the spacetime.

The above result stands for the extremal $l_1 + l_2 = l_3$ case. We would now like to extend its main features to the general non-extremal case. We are guided by the following two facts. First are the results of [20] which were done for a simpler bosonic orbifold but are found for general l's. Furthermore, as can be seen from the original discussion of [5] the important N dependence relevant to the cutoffs essentially originates from combinatorial properties of permutation group. Correspondingly, for general angular momenta $l_{1,2,3}$, we can expect the factor coming from S_N permutations of the form:

$$\frac{((N-1-l_1)!(N-1-l_2)!(N-1-l_3)!)^{1/2}}{(N-1-\frac{l_1+l_2+l_3}{2})!} . \tag{2.3}$$

It is this factor that contains the signature of non-commutativity. Firstly, one sees the "exclusion principle" that, for a non-zero correlation function, any individual angular momenta l should not exceed the bound N. There is a further bound on the sum of angular momenta characteristic of fusion rules of WZW models or of $\mathrm{SU}_q(2)$ at roots of unity. But as we will see below, the full factorial form will appear to be present in the correlations to be evaluated on a non-commutative sphere.

3. Field theory on $AdS \times S$

We will now perform two parallel calculations,one with the standard (commutative) sphere and the other with the non-commutative (fuzzy) sphere. The purpose is to directly compare the results and see that the later case exhibits the factorial terms that were featured in the orbifold calculation.

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In evaluating the correlation functions in SUGRA one has the following two-step process. The AdS dependence is projected to the boundary of the AdS space-time using the bulk to boundary propagator. For the sphere one expands in terms of spherical harmonics. The signature of non-commutative space that we are going to exhibit is associated with the S (sphere) part of this calculation. The AdS part does not lead to such a characteristic behavior and in much of what will be presented can be ignored. Indeed we should expect the essential features of the cutoff seen in the boundary correlators to be associated with the sphere part, since the models of q-deformed AdS considered in [4, 14] have the property that the boundary is commutative. Other aspects like space-time uncertainty are manifested by the non-commutative AdS part. We will in the present section summarize the calculation of correlation functions in the commuting case and then in the next section repeat the analogous calculations in the non-commutative case.

Consider for simplicity a massless scalar field with cubic interaction:

$$S = \int dx \sqrt{g} \Big((\partial\Phi)^2 + \lambda\Phi^3 + \cdots \Big) , \tag{3.1}$$

where the integral is over the $AdS \times S$ space, g is the corresponding metric and λ represents the coupling constant for the cubic interaction. In studying field theory on products $AdS \times S$ space, the wavefunctions factorize into AdS and S components

$$\Phi = \sum \Phi(\rho, t, \phi) \Psi_I(\vec{n}) ,$$

where $\Psi_I(\vec{n})$ denote the spherical functions on the sphere S. For the case S_3, one has the well-known $D^{\ell}_{mm'}(\theta, \varphi, \psi)$ functions. For three point functions, one has schematically

$$\langle \Phi\Phi\Phi \rangle_{\mathrm{AdS}} \, \langle \Psi_{\ell_1} \Psi_{\ell_2} \Psi_{\ell_3} \rangle_S \, .$$

The sphere contributions (on which we will concentrate) exhibit typically the Clebsch-Gordon coefficients

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} F\left(\ell_1, \ell_2, \ell_3; N\right)$$

and a factor $F(\ell; N)$ solely dependent on the magnitudes of angular momenta and N. In the non-commutative case, this is the structure of a star product

$$Y^{\ell_1} * Y^{\ell_2} = \sum (\text{Clebsch-Gordon}) F\left(\ell_1, \ell_2, \ell_3; N\right) Y^{\ell_3} \, .$$

The relevant information concerning non-commutativity lies in the "fusion coefficients" denoted by $F(\ell_1, \ell_2, \ell_3 : N)$. They depend solely on the magnitudes of angular momenta and the "non-commutativity" parameter N. We can expect that for studying the behavior of the fusion coefficients on these parameters, it is sufficient to look at the reduction of the sphere S_3 to S_2. Specifically one knows that the eigenfunctions on S_3

$$D^{\ell}_{mm'=0} = Y_{\ell m}(\theta, \phi)$$

reduce to spherical harmonics on S_2. This implies that the nature of non-commutativity is essentially the same for the $AdS_3 \times S_3$ space or $AdS_2 \times S_2$ which is its $\mathrm{U}(1) \times \mathrm{U}(1)$ coset. Further study of the relation between versions of non-commutative S^3 and the fuzzy sphere (or the corresponding non-compact version) would generally be important in clarifying the connection between finite N effects in CFT and non-commutative space-time. We mention that a fuzzy S_3, its riemannian geometry and its relation with string theory appears in [21].

We use the following representation for spherical harmonics:

$$Y^I = \Omega^I_{i_1 \cdots i_l} \frac{x^i_1 \cdots x^{i_l}}{\rho^l}, \tag{3.2}$$

where Ω^I is a traceless, symmetric, l-index tensor with indices $i_k = 1 \ldots 3$, x^i are the coordinates of the three-dimensional flat space and $\rho = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Y^I is an eigenvector of the sphere laplacian with eigenvalue $l(l+1)$. We also assume that the tensors Ω^I are normalized in the sense: $\Omega^I_{i_1 \cdots i_l} \Omega^J_{i_1 \cdots i_l} = \delta^{IJ}$ for equal index number and 0 otherwise. By straightforward computations presented in appendix A we obtain the following expressions for the product of two harmonics ((I,l_1), (J,l_2) indices) integrated over the sphere:

$$\langle Y^I Y^J \rangle \equiv \frac{1}{4\pi} \int_{S^2} Y^I Y^J = \frac{\pi^{1/2} \Gamma(l+1)}{2^{l+1} \Gamma(l+3/2)} \delta^{IJ}. \tag{3.3}$$

For the product of three harmonics ((I,l_1), (J,l_2), (K,l_3) indices) we obtain:

$$\begin{aligned} \langle Y^I Y^J Y^K \rangle &\equiv \frac{1}{4\pi} \int_{S^2} Y^I Y^J Y^K \\ &= \frac{\pi^{1/2} \Gamma(l_1+1) \Gamma(l_2+1) \Gamma(l_3+1)}{2^{\Sigma/2+1} \Gamma(\alpha_1+1) \Gamma(\alpha_2+1) \Gamma(\alpha_3+1) \Gamma(\frac{\Sigma+3}{2})} C^{IJK}, \end{aligned} \tag{3.4}$$

where $\Sigma = l_1 + l_2 + l_3$, $\alpha_1 = \frac{1}{2}(-l_1 + l_2 + l_3)$, $\alpha_2 = \frac{1}{2}(l_1 - l_2 + l_3)$, $\alpha_3 = \frac{1}{2}(l_1 + l_2 - l_3)$ and:

$$C^{IJK} = \Omega^I_{i_1 \cdots i_{\alpha_3} k_1 \cdots k_{\alpha_2}} \Omega^J_{i_1 \cdots i_{\alpha_3} j_1 \cdots j_{\alpha_1}} \Omega^J_{j_1 \cdots j_{\alpha_1} k_1 \cdots k_{\alpha_2}}. \tag{3.5}$$

We multiply the spherical harmonics with appropriate factors in order to normalize them. After we integrate over the sphere we obtain the following AdS_2 action:

$$S = \int_{AdS_2} d^2x \sqrt{g} \Big(\partial\phi_I \partial\phi_I + l(l+1) \phi_I \phi_I + \lambda A^{IJK} \phi_I \phi_J \phi_K \Big), \tag{3.6}$$

where g is now the AdS_2 metric and:

$$A^{IJK} = \tag{3.7}$$
$$= \frac{\pi^{-1/4}2^{1/2}\Big(\Gamma(l_1+1)\Gamma(l_1+1)\Gamma(l_3+1)\Gamma(l_1+3/2)\Gamma(l_2+3/2)\Gamma(l_3+3/2)\Big)^{1/2}}{\Gamma(\frac{\Sigma+3}{2})\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)}\,.$$

The action in AdS_2 can be related to the boundary action using the procedure and expressions developed in [3, 22] (see also [23]). The two- and three-point expressions for the constants in the correlation functions of the boundary operators O_I (corresponding to ϕ_I) are:

$$\begin{aligned}\langle O_I O_J\rangle &= \delta_{IJ}\frac{\Gamma(\Delta_I+1)}{\pi^{1/2}\Gamma(\Delta_I-1/2)}\,,\\ \langle O_I O_J O_K\rangle &= -\frac{\lambda A^{IJK}}{r2\pi}\times \\ &\quad\times\frac{\Gamma(\frac{-\Delta_I+\Delta_J+\Delta_K}{2})\Gamma(\frac{\Delta_I-\Delta_J+\Delta_K}{2})\Gamma(\frac{\Delta_I+\Delta_J-\Delta_K}{2})\Gamma(\frac{\Delta_I+\Delta_J+\Delta_K-2}{2})}{\Gamma(\Delta_I-1/2)\Gamma(\Delta_J-1/2)\Gamma(\Delta_K-1/2)}\,,\end{aligned}\tag{3.8}$$

where Δ for each operator is $l+1$. We redefine the operators such that the constant in the two-point function is δ_{IJ}. After introducing all the factors, those coming from normalization and A^{IJK}, we obtain the following expression for the three-point correlation functions:

$$\begin{aligned}\langle O_I O_J O_K\rangle = &-\frac{\lambda C^{IJK}}{(2\pi)^{1/2}}\left(\frac{(l_1+1/2)(l_2+1/2)(l_3+1/2)}{(l_1+1)(l_2+1)(l_3+1)}\right)^{1/2}\times\\ &\times\frac{\Gamma(\alpha_1+1/2)\Gamma(\alpha_2+1/2)\Gamma(\alpha_3+1/2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)}\frac{\Gamma(\frac{\Sigma-1}{2})}{\Gamma(\frac{\Sigma+3}{2})}\,.\end{aligned}\tag{3.9}$$

We observe that in our case the scalar has a mass given by the sphere laplacian $\sqrt{l(l+1)}$. In the case of $AdS_2\times S_2$ gravity, much of the analysis is in terms of chiral primary fields. The chiral primary fields are combinations of fields coming from four-dimensional gravity that have the lowest possible AdS_2 mass for a given l: $\sqrt{l(l-1)}$. We can assume that we do not start with such a simple theory as in (3.1), but with one that after sphere reduction leads to the lowest mass, appropriate for a chiral primary field [3]. For such a field, the corresponding Δ is l. In such a theory, we also consider a simple cubic interaction and obtain a qualitative picture for the sphere reduction. In this case, the three-point correlation functions for chiral primaries operators obtained in the end are simpler:

$$\langle O_I O_J O_K\rangle = -\frac{\lambda C^{IJK}}{(2\pi)^{1/2}}\frac{4\left((l_1^2-1/4)(l_2^2-1/4)(l_3^2-1/4)\right)^{1/2}}{\alpha_1\alpha_2\alpha_3(\Sigma^2-1)}\,.\tag{3.10}$$

We observe that the correlation functions in the case of chiral primary (3.10) is much simpler than (3.9). This is one of the features observed in all gravity cases of $AdS_p \times S^p$ reduction for chiral primary operators. The cancellation of factors appears between those coming from sphere and those coming from AdS spaces. We also note that (3.10) gives a divergent result for extremal cases like $l_1 + l_2 = l_3$. The divergences are due to our simplified model and they disappear in a realistic model. In the case of gravity, both the quadratic and cubic terms contain higher derivatives of the fields and these are responsible for both lower mass and consistent three-point correlation functions.

4. Field theory on non-commutative $AdS_2 \times S^2$

We will now repeat the calculation of the above section, replacing the sphere by its non-commutative counterpart. In our previous work we have argued for non-commutative (q-deformation of) space-time in the context of the AdS/CFT correspondence. The q-deformed two-sphere can be defined as a quotient of $\mathrm{SU}(2)_q$ [24] and belongs to the classification of [25]. There are also transformations between q-spheres with manifest $\mathrm{SU}(2)_q$ symmetry and spheres with manifest SU(2) symmetry. For generic q this takes the form of a connection between the classical sphere and the q-sphere. For roots of unity this takes the form of a connection between the q-sphere and the fuzzy sphere. The technical reason for these connections is essentially the deformation maps between U_q generators and the generators of the classical symmetry discussed in [26]. Applying these transformations to both the algebra of functions on the sphere and to the symmetry generators acting on the algebra gives a transformation between q-sphere with U_q symmetry and classical sphere with U(SU(2)) symmetry for generic q. This can be expected to lead, at roots of unity, to a transformation between fuzzy sphere and q-sphere. Indeed q-spheres at roots of unity are know to admit finite U_q covariant truncations [27]. The transformation is a non-commutative version of a diffeomorphism which should be a symmetry in these applications of quantum spheres to non-commutative gravity. In the following we work with the fuzzy sphere.

We first review the definition and properties of the fuzzy sphere giving formulae for the integration in parallel with the commutative case. This will lead to calculation of all the relevant quantities such as the normalization constants (3.3) and the three harmonic interaction (3.4).

The definition of the fuzzy sphere (for reviews see [28, 29, 30]) is given in terms of an algebra of polynomials in X^i, $i = 1 \ldots 3$ subject to the following constraints:

$$[X^i, X^j] = i\epsilon_{ijk} X^k \,, \qquad (X^1)^2 + \cdots + (X^3)^2 = \rho^2 \,, \tag{4.1}$$

where ρ^2 is a constant equal to $\frac{N^2-1}{4}$ (N is a positive integer measuring the fuzziness of the sphere). Such a deformation preserves the SO(3) symmetry of the sphere. We can represent the X's (and the algebra) as hermitian operators in the SO(3) representation having spin $\frac{N-1}{2}$. As such, the coordinates are now hermitian $N \times N$ matrices. In this representation we define the integral over the sphere as:

$$\frac{1}{4\pi} \int_{S^2} (\cdots) \longrightarrow \frac{1}{N} \operatorname{Tr}(\cdots)\,, \tag{4.2}$$

where Tr is the trace in the $\frac{N-1}{2}$ representation and $\cdots$ mean a function on the sphere. It is also straightforward to represent the su(2) symmetry (generators J_i) in this algebra:

$$J_i f(X) = [X^i, f(X)]\,, \quad i = 1 \ldots 3\,. \tag{4.3}$$

The spherical harmonics are constructed in the same way as in commutative sphere (3.2) replacing the commutative coordinates x^i with the noncommutative ones X^i. It is straightforward to prove that the vector space of symmetric traceless polynomial of degree l is left invariant by the su(2) generators and that it forms an irreducible representation with highest weight l.

The symmetric polynomials in X of any degree appear in the Taylor series of $\exp(iJX) = \exp(iJ_iX^i)$, where J's are regular commutative numbers. In order to compute the normalization constants and the three point interaction we construct the following quantity:

$$I(J^1, J^2, J^3) = \frac{1}{N} \operatorname{Tr}\left(e^{iJ^1X} e^{iJ^2X} e^{iJ^3X}\right). \tag{4.4}$$

We can extract from this the trace of the product of two and three symmetric polynomials as:

$$\begin{aligned}
&\frac{1}{N} \operatorname{Tr}\left(X^{(i_1} \cdots X^{i_{l_1})} X^{(j_1} \cdots X^{j_{l_2})}\right) = \partial_{J^1_{i_1}} \cdots \partial_{J^1_{i_{l_1}}} \partial_{J^2_{j_1}} \cdots \partial_{J^2_{j_{l_2}}} I(J^1, J^2, 0)|_{J^{1,2}} = 0\,,\\
&\frac{1}{N} \operatorname{Tr}\left(X^{(i_1} \cdots X^{i_{l_1})} X^{(j_1} \cdots X^{j_{l_2})} X^{(k_1} \cdots X^{k_{l_3})}\right) =\\
&\quad = \partial_{J^1_{i_1}} \cdots \partial_{J^1_{i_{l_1}}} \partial_{J^2_{j_1}} \cdots \partial_{J^2_{j_{l_2}}} \partial_{J^3_{k_1}} \cdots \partial_{J^3_{k_{l_3}}} I(J^1, J^2, J^3)|_{J^{123}=0}\,,
\end{aligned} \tag{4.5}$$

where $(\cdots)$ means the symmetrized product of X's.

The evaluation of $I(J^1, J^2, J^3)$ can be done if we note that the r.h.s. of (4.4) is the trace of a product of three SO(3) rotations with parameters $J^{1,2,3}$. The product of three rotations is itself a rotation with a parameter $J = J(J_1, J_2, J_3)$:

$$e^{iJX} = e^{iJ^1X} e^{iJ^2X} e^{iJ^3X}\,, \tag{4.6}$$

and the trace of this operator can be evaluated easily in a basis where JX is diagonal as:

$$I(J) \equiv I(J^1, J^2, J^3) = \frac{1}{N} \frac{\sin(JN/2)}{\sin(J/2)}\,. \tag{4.7}$$

The dependence of J (or rather $\cos(J/2)$) on $J^{1,2,3}$ can be easy computed (see appendix B) and we list here the result:

$$\begin{aligned}
&\cos\left(\frac{J}{2}\right) = \\
&= \cos\left(\frac{J^1}{2}\right)\cos\left(\frac{J^2}{2}\right)\cos\left(\frac{J^3}{2}\right) - \cos\left(\frac{J^1}{2}\right)\frac{\sin(J^2/2)}{J^2}\frac{\sin(J^3/2)}{J^3}(J^2J^3) - \\
&- \cos\left(\frac{J^2}{2}\right)\frac{\sin(J^3/2)}{J^3}\frac{\sin(J^1/2)}{J^1}(J^1J^3) - \cos\left(\frac{J^3}{2}\right)\frac{\sin(J^1/2)}{J^1}\frac{\sin(J^2/2)}{J^2}(J^1J^2) + \\
&+ \frac{\sin(J^1/2)}{J^1}\frac{\sin(J^2/2)}{J^2}\frac{\sin(J^3/2)}{J^3}(J^1 \times J^2)J^3\,. \qquad (4.8)
\end{aligned}$$

The cubic interaction $\operatorname{tr}\Phi^3$ on a fuzzy sphere leads, when we expand the field $\Phi = \sum_I a_I \hat{Y}_I$, to a coefficient of $a_I a_J a_K$ which is the symmetrized sum:

$$\frac{1}{N}\operatorname{Tr}(Y^I Y^J Y^K) + \frac{1}{N}\operatorname{Tr}(Y^I Y^K Y^J) \qquad (4.9)$$

and other terms related by cyclicity of the trace. This coefficient is what appears in the computations in field theory on AdS_2 as outlined in section 3. The symmetrization amounts in the end to dropping the last factor appearing in the expression of $\cos(J/2)$ (4.8), the only one not symmetric in $J^{1,2,3}$. After this change, we are left with J depending on the following variables only: $|J^1|$, $|J^2|$, $|J^3|$, J^1J^2, J^2J^3 and J^1J^3, where $|J| = \sqrt{J^2}$.

The method used for the evaluation of the two- and three-interaction for harmonics is given in (4.5). Spherical harmonics come with polynomial in X's that are both symmetric and traceless. The traceless property of polynomials is shifted to the traceless of partial derivatives in J^1, J^2 and J^3 (4.5) and as such, leads after setting J's to 0 to the following expression (we denote by $\tilde{A}^{(i)(j)}$ and $\tilde{A}^{(i)(j)(k)}$ the l.h.s. of eqs. (4.5) with the property that the indices i's, j's and k's are also traceless):

$$\begin{aligned}
\tilde{A}^{(i)(j)} &= \delta_{l_1 l_2}\frac{1}{2^{2l_1}}\left(\frac{\partial}{\partial(\cos(J/2))}\right)^{l_1} I(J)|_{J=0}(\delta^{i_1 j_1}\cdots\delta^{i_{l_1} j_{l_1}} + \cdots)\,, \qquad (4.10)\\
\tilde{A}^{(i)(j)(k)} &= \frac{1}{2^{l_1+l_2+l_3}}\left(\frac{\partial}{\partial(\cos(J/2))}\right)^{\frac{l_1+l_2+l_3}{2}} I(J)|_{J=0}(\delta^{i_1 j_1}\cdots\delta^{i_{\alpha_3+1} k_1}\cdots + \cdots)\,.
\end{aligned}$$

The $\cdots$ in the last equations mean all possible contractions between indices i's and j's, j's and k's and i's and k's. We have also to specify:

$$\left(\frac{\partial}{\partial(\cos(J/2))}\right)^{l} I(J)|_{J=0} = \frac{(N+l)!}{N(N-1-l)!(2l+1)!!}\,, \qquad (4.11)$$

and we give a derivation for this in the appendix. The final results for harmonics can be written in terms of those obtained in the commutative case (we denote the corresponding terms for the fuzzy sphere as $\langle Y^I Y^J\rangle_N$ and $\langle Y^I Y^J Y^K\rangle_N$):

$$\begin{aligned}
\langle Y^I Y^J\rangle_N &= \frac{(N+l_1)!}{(2\rho)^{2l_1} N(N-l_1-1)!}\langle Y^I Y^J\rangle\,,\\
\langle Y^I Y^J Y^K\rangle_N &= \frac{(N+\frac{l_1+l_2+l_3}{2})!}{(2\rho)^{l_1+l_2+l_3} N(N-1-\frac{l_1+l_2+l_3}{2})!}\langle Y^I Y^J Y^K\rangle\,,
\end{aligned} \tag{4.12}$$

where ρ is given (4.1). In the $N\to\infty$ limit, the factors coming from the fuzzy sphere go to 1 and the results for the commutative sphere are obtained.

We have now prepared all the necessary ingredients to proceed to our main calculations and evaluate correlation functions associated with gravity on $AdS_2\times S^2_{\text{fuzzy}}$.

Consider now the following action:

$$S=\int_{AdS_2} d^2x\sqrt{g}\frac{1}{N}\,\mathrm{Tr}(\partial\phi\partial\phi + J_i\phi J_i\phi+\lambda\phi^3)\,, \tag{4.13}$$

where the integration over a (commutative) sphere is replaced by the Tr over symmetric product of functions defined on the non-commutative (fuzzy) sphere. Expanding in terms of associated spherical harmonics, we obtain the same action on AdS_2 as in (4.13) but with A^{IJK} replaced by:

$$\begin{aligned}
A_N^{IJK} =& \frac{((N-1-l_1)!(N-1-l_2)!(N-1-l_3)!)^{1/2}}{(N-1-\frac{l_1+l_2+l_3}{2})!}\times\\
&\times\frac{N^{1/2}(N+\frac{l_1+l_2+l_3}{2})!}{((N+l_1)!(N+l_2)!(N+l_3)!)^{1/2}}A^{IJK}\,.
\end{aligned} \tag{4.14}$$

After we reduce the theory on AdS_2 in the same way as before we obtain the result. The correlation functions in the case of non-commutative sphere are equal with those in the case of commutative sphere multiplied with the same factor as in (4.14). We give below the result for the chiral primaries correlation functions in this case (see (3.10)):

$$\begin{aligned}
\langle O_I O_J O_K\rangle_N =& -\frac{\lambda C^{IJK}}{(2\pi)^{1/2}}\frac{((N-1-l_1)!(N-1-l_2)!(N-1-l_3)!)^{1/2}}{(N-1-\frac{l_1+l_2+l_3}{2})!}\times\\
&\times\frac{N^{1/2}(N+\frac{l_1+l_2+l_3}{2})!}{((N+l_1)!(N+l_2)!(N+l_3)!)^{1/2}}\frac{4((l_1^2-1/4)(l_2^2-1/4)(l_3^2-1/4))^{1/2}}{\alpha_1\alpha_2\alpha_3(\Sigma^2-1)}\,.
\end{aligned} \tag{4.15}$$

This expression contains the essential features relevant for comparison with the CFT. One has the SU(2) symmetry, that was represented as R-symmetry in CFT. The result exhibits a cutoff at $l_{1,2,3}=N$ and also at $\frac{l_1+l_2+l_3}{2}=N$. Most significantly the overall factor that we see in the non-commutative (fuzzy) sphere result is of identical form to the corresponding factor given by the S_N orbifold . This is clear evidence for non-commutativity in AdS/CFT. The characteristic features of the factorial terms describing this non-commutativity are seen to be captured by the sphere. It is relevant to stress that the AdS contribution is not of the same form, which is understandable as we argued at the beginning of section 3 in terms of the commutative nature of the boundary in deformed AdS. It should be stated that the above expressions are not exactly equal in form to those of the finite-N CFT. First the computation of this section is done for a simplified model. In addition, the gravitational coupling being given by $1/N$ means that loop effects will also contribute to the final-N dependence. Finally deformations of AdS will also imply corresponding N-dependence. But we expect that these two effects, while providing N-dependence, do not lead to a contribution of the form that we have identified and associated with a fuzzy sphere.

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A. Spherical harmonics

In this appendix we give the formulas and some derivations used in section 3. For the computations for harmonics we use the following expressions:

$$\begin{aligned}\frac{1}{4\pi}\int_{S^2}\frac{x^{i_1}\cdots x^{i_l}}{\rho^l} &= \frac{\partial_{J_{i_1}}\cdots\partial_{J_{i_l}}\int d^3x e^{-\frac{1}{2}x^2+Jx}|_{J=0}}{4\pi\int_0^\infty d\rho\rho^{l+2}e^{-\frac{1}{2}\rho^2}} \\ &= \frac{\pi^{1/2}}{2^{\frac{l+2}{2}}\Gamma(\frac{l+3}{2})}(\delta^{i_1 i_2}\cdots+\cdots)\,, \end{aligned}\tag{A.1}$$

where $\cdots$ mean all possible contractions between i's. Then we obtain (3.3) from:

$$\langle Y^I Y^J\rangle = \Omega^I_{i_1\cdots i_{l_1}}\Omega^J_{j_1\cdots j_{l_1}}\frac{1}{4\pi}\int_S^2\frac{x^{i_1}\cdots x^{i_l}x^{j_1}\cdots x^{j_{l_2}}}{\rho^{l_1+l_2}}\tag{A.2}$$

and in a similar fashion (3.4). The extra combinatorial factors as $l!$ for (3.3) and $\alpha_1!\alpha_2!\alpha_3!$ and $\Sigma!$ come from different combinatorial way to match the indices for Ω's.

B. Non-commutative S^2

In this appendix, we derive the formulas used in section 4. In order to derive the expression (4.8) we use the spin 1/2 representation for SO(3) rotations $e^{iJ^{1,2,3}x}$, replacing $x \to \sigma/2$ with σ being the Pauli matrices, and we obtain:

$$\cos\left(\frac{J}{2}\right) + i\frac{\sin(J/2)}{J}J\sigma = \prod_{k=1}^{3}\left(\cos\left(\frac{J^k}{2}\right) + i\frac{\sin(J^k/2)}{J^k}J^k\sigma\right). \tag{B.1}$$

From this, after straightforward algebraic manipulations we obtain (4.8).

For (4.11), we define:

$$K^l_{N+1} \equiv \left(\frac{d}{d(\cos J)}\right)^l \left(\frac{\sin(J(N+1))}{\sin J}\right)\Big|_{J=0} \tag{B.2}$$

and using $\sin(J(N+1)) = \cos(JN)\sin J + \sin(JN)\cos J$ we obtain the following recurrence relations and conditions:

$$K^l_{N+1} = K^l_N + (N+l)K^{l-1}_N\,, \quad l \geq 1\,, \qquad K^0_N = N\,, \qquad K^l_1 = 0\,, \quad l \geq 1\,. \tag{B.3}$$

It is now easy to see that the expression in the r.h.s. of (4.11) satisfies (B.3).

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